Minimal Model Program

Learning Seminar.

Week 9:

- Terminal 3-fold MMP
- Terminalizations.
- Small $\mathbb{Q}$-factorializations.
- Toric singularities.

Terminal 3-fold MMP:

X terminal projective 3-fold.

$$\begin{tikzcd} X \arrow[rr, dashed, "\pi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & W & \end{tikzcd}$$

is a flipping contraction.

X $\mathbb{Q}$-factorial.
$\rho(X/W) = 1$
$-K_X$ ample over W
X has terminal sing.
X is smooth in cod 2.

What we want: construct π an isom in cod 1.

$\rho(X^+/W) = 1$
K_{X^+} is ample over W
X^+ has terminal sing.
X^+ $\mathbb{Q}$-factorial.

$$\begin{tikzcd} X \arrow[rr, dashed, "\pi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & W & \end{tikzcd}$$

X^+ is unique and coincides with

$$\mathrm{Proj}_W \bigoplus_{n \geq 0} f_* \mathcal{O}_X(nK_X)$$

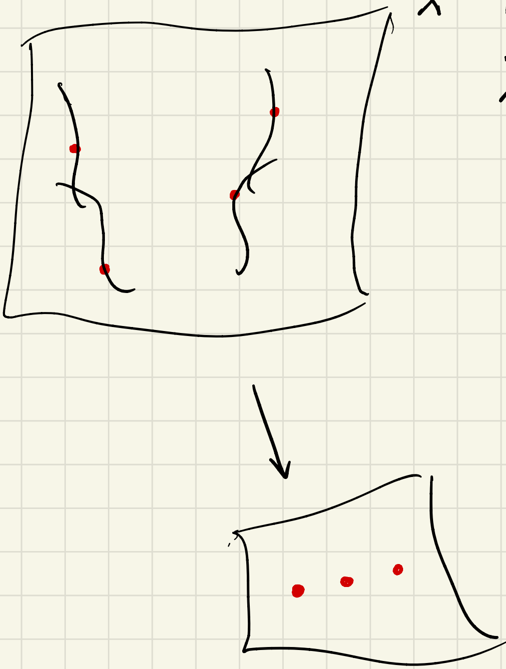
$$R(X) = \bigoplus_{n \geq 0} f_* \mathcal{O}_X(nK_X)$$

provided that this ring is a f.g $\mathcal{O}_W$-algebra

Proposition: $R(X)$ is f.g as an $\mathcal{O}_W$-algebra iff it is f.g locally over W. (even locally analytically on W).

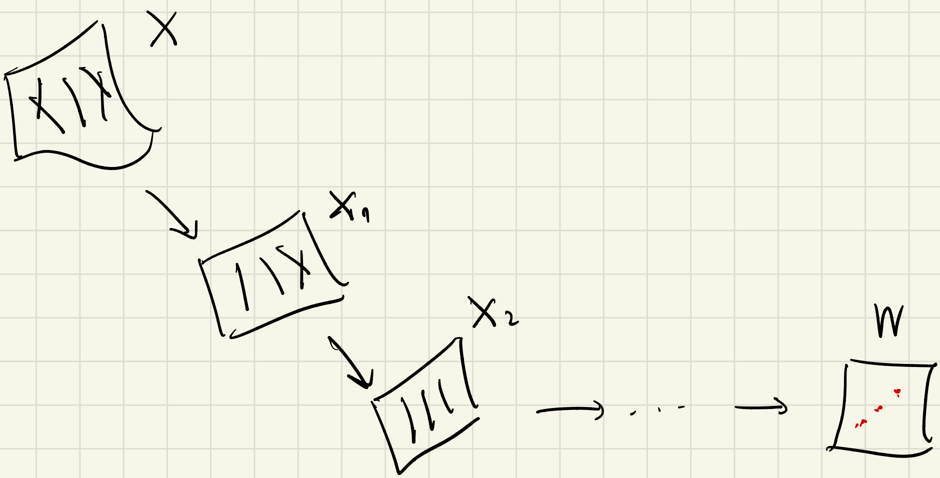
X terminal 3-fold.

X $\mathbb{Q}$-factorial.

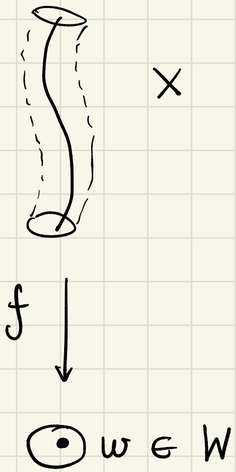


W is not $\mathbb{Q}$-factorial.

Mori 1988: proved that these curves can be contracted one by one in the analytic cat.



Terminal 3-fold flipping contr $\longrightarrow$ extremal neighborhoods.



X terminal 3-fold

$w \in W$ closed point.

$f^{-1}(w) = \mathbb{P}^1$

$-K_X$ ample over W.

(W, w) is a rational sing

$\rho(X/W) = 1$.

Question: What happens if we have a smooth extremal neighborhood?

Smooth extremal neighborhoods:

Prop: Let $X \supseteq C \simeq \mathbb{P}^1$ be an extremal neighborhood. Then $\mathcal{O}_C(K_X) \simeq \mathcal{O}_C(-1)$, $I_C/I_C^2 \simeq \mathcal{O}_C \oplus \mathcal{O}_C(1)$ and $|-K_X|$ has a smooth member.

(I_C is the ideal sheaf of C on X).

Proof: $K_X.C = -1$, from $K_X.C < 0$ and $H^1(\mathcal{O}_C(K_X)) = 0$.

$$0 \longrightarrow \mathcal{I}_C/\mathcal{I}_C^2 \longrightarrow \Omega_X^1 \otimes \mathcal{O}_C \longrightarrow \mathcal{O}(K_C) \longrightarrow 0.$$

we deduce that there is an isomorphism

$$\Lambda^2(I_C/\mathcal{I}_C^2) \xrightarrow{\simeq} \mathcal{O}(K_X) \otimes \mathcal{O}_C(-K_C)$$

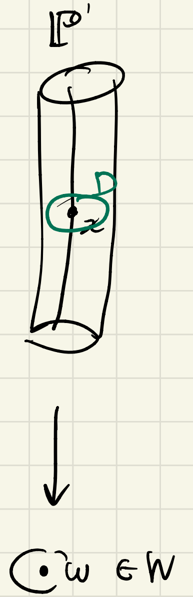
Taking degree, we conclude

$$\deg(I_C/I_C^2) = (K_X.C) - \deg K_C = 1.$$

$$0 \to I_C/I_C^2 \otimes \mathcal{O}_C(-1) \longrightarrow \mathcal{O}_X/\mathcal{I}_C^2 \otimes \mathcal{O}_X(K_X) \longrightarrow \mathcal{O}_C(-1) \longrightarrow 0.$$

$H^1(I_C/I_C^2 \otimes \mathcal{O}_C(-1)) = 0$. Hence,

$$\mathcal{I}/\mathcal{I}_C^2 \otimes \mathcal{O}_C(-1) \simeq \mathcal{O}_C \oplus \mathcal{O}_C(-1).$$



$x \in C$, (D, x) smooth div on the germ (X, x), D extends naturally to a divisor D' of X.

$$D' \in |-K_X| \,?$$

$\mathrm{Pic}\, X \simeq \mathbb{Z}$, and the isomorphism is induced by

$$\begin{array}{ccc} \mathrm{Pic}\, X & \longrightarrow & \mathbb{Z} \\ \cup & & \cup \\ \mathcal{L} & \longmapsto & \mathcal{L}\cdot C \\ & & \text{is} \\ & & \mathbb{P}^1 \end{array}$$

$D' \cdot C = 1$

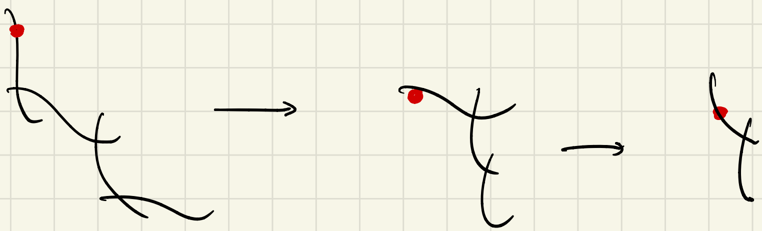
$C \cdot K_X = -1$

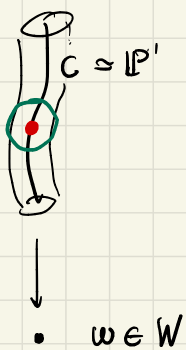
$$D' \sim -K_X.$$

$\square$

Corollary: In the above case, X is the blow-up of a smooth 3-fold (W, w) along a smooth curve C_0 passing through w.

Corollary: Let $X \xrightarrow{f} W$ be a terminal 3-fold flipping contraction. For every $w \in W$, $f^{-1}(w)$ contains a singularity.





Classified terminal 3-fold sing.

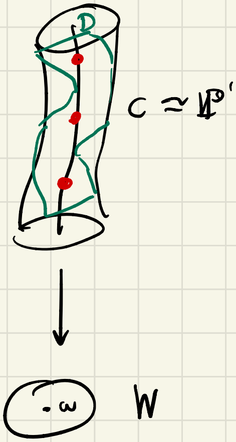
1.- There are at most 3 singular points.

2.- There is some nice divisor D passing through some of these sing points.

Theorem (Mori, 88): Let $X \supseteq C \simeq \mathbb{P}^1$ be an extremal nbd. Then one of the following on the linear systems $|-aK_X|$ $(a = 1$ or $2)$ holds:

i) $|-K_X|$ has a member D with Du Val sing, or

ii) $|-2K_X|$ has a member D so that the double over Z of X branched locus D has only Du Val sing.



D being Du Val $\Longrightarrow$ (X, D) is purely log terminal.

The finite generation of

$$\bigoplus_{n \geq 0} f_* \mathcal{O}_X(nK_X)$$

$\Longleftarrow$ The finite generation of

$$\bigoplus_{n \geq 0} f|_D \, \mathcal{O}_D(nK_X).$$

The latter is a problem about projective surfaces. $\square$

This concludes the existence of flips for terminal 3-folds

Term 3-fold.

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr] & & X_1 \arrow[dl] \arrow[r] & X_2 \arrow[rr, dashed] \arrow[dr] & & X_3 \arrow[dl] \arrow[r, dashed] & \cdots \\
& W_1 & & & W_3 & & \cdots
\end{tikzcd}$$

Theorem: An arbitrary sequence of 3-dim extremal canonical $(K_X+\Delta)$-flips is finite

Lemma: Let $\phi : X \dashrightarrow X'$ be a $(K_X+\Delta)$-flip of a 3-dim canonical pair $(X, \Delta = \sum_{i=1}^{k} a_i D_i)$.
Let $C' \subseteq X'$ be a flipped curve, and $E_{C'}$ be the exceptional divisor obtained by blowing up C'.
Then X' is smooth along C' and
generic point

$$0 \leq a(E_{C'}, X, \Delta) < a(E_{C'}, X', \Delta') = 1 - \sum \overset{1}{\cancel{a_i}} \, \mathrm{mult}_{C'}(D_i')$$

where $\mathrm{mult}_{C'}(D')$ is the multiplicity of D' along C'.

Proof: Since C' is a flipped curve, then X' is smooth along the generic pt of C'. Indeed X' is terminal along $\eta_{C'}$, so smooth.
If there is a non-terminal val with center on C', then there is a non-canonical val on (X, Δ).

Difficulty function: $(X, \Delta = \sum_i a_i D_i)$ canonical pair with D_i pairwise diff prime divisors $\alpha = \max\{a_i\}$.

$S := \sum_i a_i \mathbb{Z}_{\geq 0} \subseteq \mathbb{Q}$. We set

$$d(X,\Delta) = \sum_{\substack{x \in S \\ x \geq \alpha}} \# \left\{ \begin{array}{l} \text{Exceptional divisors over } X \\ \text{with } a(E, X, \Delta) < 1 - x \end{array} \right\}.$$

$x = 0$.

Ex: cA_r singularities. ; $d(cA_r) = r$.

Rm: The diff function is measuring # of non-term val

$d(X,\Delta) < \infty$ and $d(X,\Delta)$ does not increase

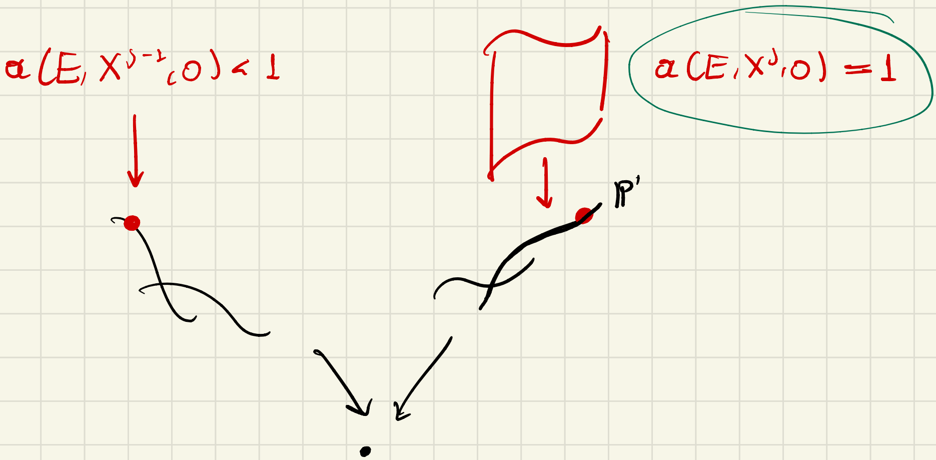
after a flip.

Termination of canonical 3-fold flips:

Proof: $\Delta = \sum_{i=1}^{k} a_i D_i$, $a_1 \le \cdots \le a_k$.

$$(X, \Delta) \overset{\phi^1}{\dashrightarrow} (X^1, \Delta^1) \overset{\phi^2}{\dashrightarrow} (X^2, \Delta^2) \overset{\phi^3}{\dashrightarrow} \cdots$$

If $k=0$, then $d(X^{j-1}, 0) > d(X^j, 0)$.



Hence, after finitely flips $d(X^\ell, 0) = 0$ and then there is no more flips.

Assume $k \ge 0$. $d(X^j, \Delta^j)$ is non-decreasing.

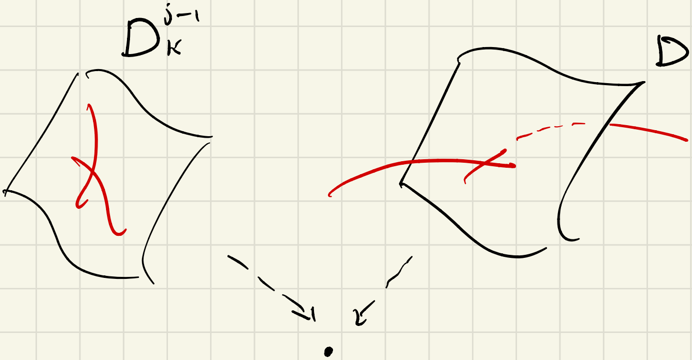
C^j flipped curve for ϕ^{j-1} assume is contained in D_k^j.

then $a_k < 1$ and $d(X^{j-1}, \Delta^{j-1}) > d(X^j, \Delta^j)$.

Thus, for $j \gg 0$, D_k^j contains no flipped curves.

Denote by $\bar{D}_K^j$ the normalization of D_K^j.

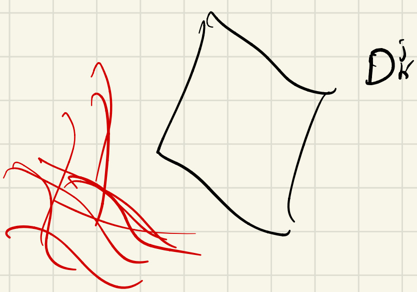
$\bar{D}_K^{j-1} \longrightarrow \bar{D}_K^j$ is a birational morphism.



The exc curves of $\bar{D}_K^j \longrightarrow \bar{D}_K^\ell$ for $\ell > K$ are l.i.

At some point we have $\bar{D}_K^j \simeq \bar{D}_K^\ell$ for $\ell \gg j$.

This means that both the flipping and flipped curves are disjoint from D_K^j.



$C \cdot D_K^j = 0$

$(X, \Delta = \sum_{i=1}^{k} a_i D_i)$ flips

$\Downarrow$

$(X, \Delta' = \sum_{i=1}^{k-1} a_i D_i)$ flips

By induction on K, these flips stop $\square$

Abundance: If X is klt + K_X nef $\Longrightarrow$ K_X semiample.

This is proved for terminal 3-folds by Kawamata.

These three results settles down the MMP for terminal 3-folds.

Existence of flips ✓ — 1988 Mori

Term of flips ✓ — 1988 Mori. Kollár – Shokurov.

Abundance ✓ — Kawamata 90's.

Existence of flips:

2000's Kollár – Shokurov.
from (X,Δ) canonical to (X,Δ) lc 3-folds.

2005: Hacon & Mckernan:
flips exists in dim n
provided the MMP works in dim $n-1$.

2006: BCHM existence of flips
(X,Δ) klt.

2010: HX-Bir: existence of flips
(X,Δ) lc.

Termination of flips:

2004: Alexeev – Hacon – Kawamata
term of flips for (X,Δ) 4-fold.
$-(K_X+\Delta)$ eff.

2000: Fujino proved term of flips for (X,Δ) a terminal 4-fold.

2006: BCHM term of flips
(X,Δ) klt $K_X+\Delta$ big.

2018: Term of flips for (X,Δ) lc 4-fold with $K_X+\Delta$ pseff.

(X,Δ) klt 4-fold uniruled
(X,Δ) dim ≥ 5.
unknown

Applications to singularities:

Conjecturally, the MMP contracts / flips the locus:

$$Bs(K_X) = Bs_-(K_X)$$

This is known in dim 3 and it follows from termination + abundance

Recall: $D \subseteq X$, A ample divisor on X.

$$Bs_-(D) = \bigcup_{\varepsilon > 0} Bs(D + \varepsilon A) \subseteq Bs(D)$$

Countable union of alg varieties.

Leis ... showed the existence of a divisor on certain blow-up of $\mathbb{P}^3$ whose Bs_- is a countable union of curves.

Terminalization: Let (X, Δ) be a klt pair of dim 3. Then there exists a projective birational morphism $Y \longrightarrow X$ so that Y is terminal and extracts exactly the divisors with $a_E(X, \Delta) \in (-1, 0]$.

Proof.

$$\begin{array}{c} (Z, \Delta_Z) \\ \big\downarrow \varphi \\ (X, \Delta) \end{array}$$

$\varphi^*(K_X + \Delta) = K_Z + \Delta_Z$ not 2 bnd

Δ_Z' is Δ_Z after we increase all neg coeff to 0.

Proof.

$$\begin{tikzcd}
(Z, \Delta_Z) \arrow[d, "\varphi"'] \arrow[dr, dashed] & \\
(X, \Delta) & (Y, \Delta_Y) \arrow[l]
\end{tikzcd}$$

$\varphi^*(K_X + \Delta) = K_Z + \Delta_Z$ not 2 but

Δ_Z' is Δ_Z after we increase all neg coeff to 0.

$$Bs_-(K_Z + \Delta_Z') \supseteq \text{All divisors with } a_E(X, \Delta) > 0.$$

R the MMP for $K_Z + \Delta_Z'$. we contract all these divisors

(Z, Δ_Z') terminal $\Longrightarrow$ when you run the MMP it remains terminal.

$\square$.

Abundance: lc pairs (X, Δ) dim 3 ✓✓

k pairs (X, Δ) dim 4 with X uniruled ✓✓

lc pair (X, Δ) dim 4 with $K_X + \Delta$ pseff ??

Small $\mathbb{Q}$-factorialization: Let (X, Δ) be a klt pair of dim 3.
Then there exists a projective birational morphism $Y \xrightarrow{\pi} X$ so that π is a small morphism (does not extract divisors). and Y is $\mathbb{Q}$-factorial. In particular, $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$ defines a klt pair (Y, Δ_Y) and Y is klt.

Sketch:

$$\begin{array}{c} (Z, \Delta_Z) \\ \varphi \big\downarrow \\ (X, \Delta) \end{array}$$

a log resolution of (X, Δ)

$$\varphi^*(K_X + \Delta) = K_Z + \Delta_Z.$$

Δ_Z may have negative coefficients, (Z, Δ_Z) is a sub-klt pair.
Let $\varepsilon > 0$ so that all coefficents of Δ_Z are less then $1 - \varepsilon$.
This ε exists by the klt-ness assumption
Let Δ_Z' be the divisor obtained from Δ_Z by increasing all the coeff of exc divisors over X to $1 - \varepsilon$
Then, by the negativity lemma, we obtain

$$\operatorname{supp}(Ex(Z/X)) \subseteq Bs_-(K_Z + \Delta_Z' / X).$$

i.e., the diminished base locus of $K_Z + \Delta'_Z$ over X contains the exceptional locus of $Z \longrightarrow X$, which we may assume purely divisorial.

Hence, when run the MMP for $K_Z + \Delta'_Z$ relative over X:

$$\begin{tikzcd}
(Z, \Delta'_Z) \arrow[r, dashed] \arrow[d, "\varphi"'] & (Z_1, \Delta^1_{Z_1}) \arrow[r, dashed] \arrow[dl, "\varphi_1"] & \cdots \arrow[r, dashed] & (Z_K, \Delta'_{Z_K}) \arrow[dlll, "\varphi_K"] \\
(X, \Delta) & & &
\end{tikzcd}$$

All the divisors of $Ex(Z/X)$ are contracted. (*) The MMP terminates because we are working in dimension 3. We call $K_{Z_K} + \Delta'_{Z_K}$ the last model of this MMP. Since $K_Z + \Delta'_Z$ is big over X, then $K_{Z_K} + \Delta'_{Z_K}$ is big and nef over X.

Furthermore, Z_K is $\mathbb{Q}$-factorial, since Z is $\mathbb{Q}$-fact and the MMP preserves $\mathbb{Q}$-factoriality.

By (*) the morphism φ_K is small. Hence $\varphi_K^*(K_X + \Delta) = K_{Z_K} + \Delta'_{Z_K}$. We can set $Y = Z_K$ and conclude the proof. $\square$

Dlt modification: Let (X,Δ) be a log canonical pair (of dim 3)

There exists a projective birational morphism $\pi : Y \longrightarrow X$ so that it only extract divisors E so that $a_E(X,\Delta)=0$ and $K_Y+\Delta_Y=\pi^*(K_X+\Delta)$ defines a dlt pair (Y,Δ_Y)

Sketch: Let

$$\begin{array}{c}(Z,\Delta_Z)\\ \Big\downarrow \varphi \\ (X,\Delta)\end{array}$$

be a log resolution of (X,Δ).

We define Δ_Z' to be the divisor obtained from Δ_Z by increasing to 1 all coefficients from the prime components of Δ_Z which are exceptional over X and have coeff <1.

By the negativity Lemma, we have:

$$\bigcup_{\substack{E \text{ exc over } X \\ a_E(X,\Delta)>0}} \operatorname{supp}(E) \subseteq \mathbf{B}_{-}(K_Z+\Delta_Z'/X).$$

By the log smoothness, (Z,Δ_Z') is dlt.

We run a MMP for $K_Z+\Delta'_Z$ over X.

We call (Z_K, Δ'_{Z_K}) the last model of this minimal model program.

It contracts all the divisors on Z which are exceptional over X and satisfy $a_E(X,\Delta)\geq 0$.

Hence, $Z_K \xrightarrow{\pi} X$ only extract divisors with $a_E(Z_K, \Delta'_{Z_K}) = 0$.

Since the MMP preserves the dlt property, then we have that (Z_K, Δ'_{Z_K}) is dlt.

Hence, it suffices to take $Y = Z_K$.

The following is a corollary of existence of small $\mathbb{Q}$-fact.

Corollary: A klt surface sing is $\mathbb{Q}$-factorial.

Proof: A small $\mathbb{Q}$-fact is in this case

Remark:

- The existence of small $\mathbb{Q}$-fact in dimension n follows from the existence and termination of flips for klt pairs (X,Δ) with $K_X+\Delta$ big over the base in dimension n.

- The existence of terminalizations in dimension n follows from the existence and termination of flips for klt pairs (X,Δ) with $K_X+\Delta$ big over the base in dimension n.

- The existence of dlt modification in dimension n follows from the existence and termination of flips for dlt pairs (X,Δ) with $K_X+\Delta$ big over the base in dimension n.

However, in a paper by Kollár and Kovács, there is a proof (due to Hacon) only using MMP for klt pairs.

Toric singularities:

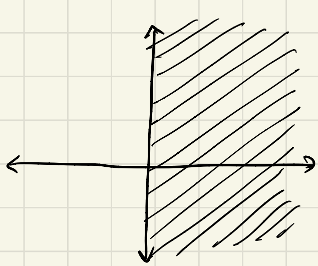
Toric geometry is the side of algebraic geometry that comes from combinatorics. The equation defining toric varieties and toric singularities are binomial equations and these binomial equations are encrypted by certain convex bodies.

Let N be a free finitely generated abelian group and $M = \mathrm{Hom}(N, \mathbb{Z})$ its dual.

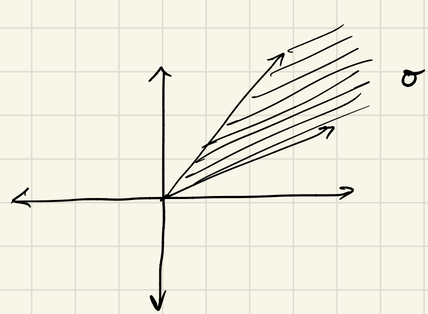
$N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ the associated $\mathbb{Q}$-vector spaces

Let $\sigma \subseteq N_{\mathbb{Q}}$ be a strictly convex polyhedral cone.

strictly convex means that it doesn't contain linear subs.



convex polyhedral.
not strictly convex



strictly convex polyhedral cone.

Given $\sigma \subseteq N_{\mathbb{Q}}$ strictly convex polyhedral cone.

$$\sigma^{\vee} = \{ u \in M_{\mathbb{Q}} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma \}.$$

$\sigma^{\vee}$ is also a strictly convex polyhedral cone

$\mathbb{K}[\sigma^{\vee} \cap M]$ is the correspondng ring associated to the semigroup $\sigma^{\vee} \cap M$

We define $X(\sigma) := \operatorname{Spec}(\mathbb{K}[\sigma^{\vee} \cap M])$.

Example:

$$\sigma = \operatorname{span}\{(-n,1), (n,1)\} \subseteq \mathbb{Q}^2.$$

$$\sigma^{\vee} = \operatorname{span}\{(1,n), (-1,n)\} \subseteq \mathbb{Q}^2.$$

The semigroup $\sigma^{\vee} \cap M$ is generated by

$$\underset{x}{\underset{\parallel}{(1,n)}}, \quad \underset{y}{\underset{\parallel}{(-1,n)}} \quad \text{and} \quad \underset{z}{\underset{\parallel}{(0,1)}}$$

with the relation $(1,n) + (-1,n) = 2n\,(0,1)$

Hence, $X(\sigma) \simeq \mathbb{K}[x,y,z] / (xy - z^{2n})$.

Toric geometry:

The M-grading on $\mathbb{K}[\sigma^\vee \cap M]$ induces a

$\operatorname{Spec}(\mathbb{K}[M]) \simeq \mathbb{G}_m^{\dim_{\mathbb{Q}}(M_{\mathbb{Q}})}$ - action on $X(\sigma)$

Let's set $n = \dim(X(\sigma)) = \dim_{\mathbb{Q}}(M_{\mathbb{Q}})$

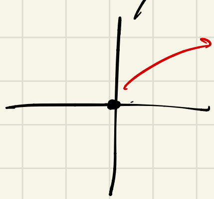
$X(\sigma)$ can be decomposed in $\mathbb{G}_m^n$-orbits, so that a $\mathbb{G}_m^{n-\ell}$-orbit corresponds to a ℓ-dimensional face of σ.

Example: σ as in previous example.

$\mathbb{K}[x,y,z]/\langle xy - z^{2n}\rangle$

If $z = 0$, we obtain

$\operatorname{spec} \mathbb{K}[x,y]/\langle x,y\rangle =$



$\to \mathbb{G}_m^1$ corresponds to one ray of σ

$\mathbb{G}_m^0$

$\to \mathbb{G}_m$ corresponds to the other ray

If $z \neq 0$, then $x \neq 0$, $y \neq 0$

and $(t_1, t_2) \longmapsto (t_1, t_1^{-1}t_2^{2n}, t_2)$.

gives an iso of $\mathbb{G}_m^2$ with this chart

$\mathbb{Q}$-factorial and smooth toric points:

The affine toric variety $X(\sigma)$ is smooth iff σ is a regular cone of $M_{\mathbb{Q}}$, i.e., its extremal rays span M (over $\mathbb{Z}$).

If σ is regular, then $X(\sigma) \simeq \mathbb{C}^n$.

The affine toric variety $X(\sigma)$ is $\mathbb{Q}$-factorial iff σ is simplicial in $M_{\mathbb{Q}}$, i.e., its extremal rays span $M_{\mathbb{Q}}$ (over $\mathbb{Q}$).

If σ is simplicial, then $X(\sigma) \simeq \mathbb{C}^n/A$, where A is a finite abelian group acting monomially on $\mathbb{C}^n$.

Example: $\sigma = \text{span}\{(-n,1), (n,1)\} \subseteq \mathbb{Q}^2$ defines $X(\sigma)$ which is $\simeq \mathbb{C}^2/\mu_{2n} \rightarrow$ 2n root of unity.

Let $\sigma = \text{span}\{(1,0,0), (0,1,0), (0,0,1), (1,-1,1)\} \subseteq \mathbb{Q}^3$.

Then $X(\sigma)$ is isomorphic to a cone over $\mathbb{P}^1 \times \mathbb{P}^1$.

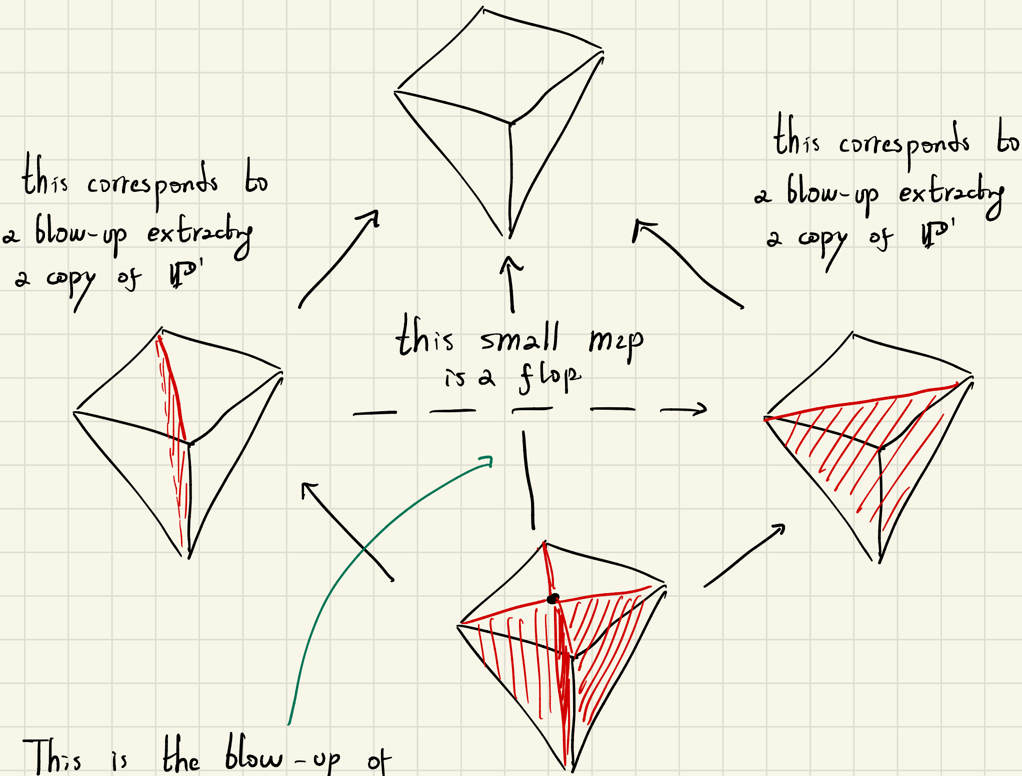
Thus, is not $\mathbb{Q}$-factorial (its local class group contains a copy of $\mathbb{Z}$).

Small $\mathbb{Q}$-fact of toric singularity:

A small $\mathbb{Q}$-factorialization of a toric sing corresponds to a simplicialization of σ (cone refinement). That does not introduce new rays.

Example:

$$\sigma = \operatorname{span}\{(1,0,0),\ (0,1,0),\ (0,0,1),\ (1,-1,1)\} \subseteq \mathbb{Q}^3$$



This is the blow-up of the maximal ideal at the vertex is a (not-small) $\mathbb{Q}$-factorialization.